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On Some Aspects of Generalized Discrete Pareto Distribution

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ABSTRACT
Here we introduce generalized discrete Pareto distribution as a flexible model to analyse count data, and includes as special cases: the discrete Burr, discrete Pareto, discrete generalized Pareto and alternative discrete Pareto distributions. We obtain limiting behaviour of parameters, infinite divisibility, moments and their bounds, hazard rates, stochastic orderings and aging intensity properties. The method of maximum likelihood estimation is used herein to estimate model parameters. Simulation study is carried out to examine the bias, mean square error, average absolute bias and mean relative error of maximum likelihood estimators. Finally, the paper illustrates the flexibility of the proposed distribution using medical dataset, demonstrating its superior fit compared to other models.

KEYWORDS
Characterizations; Count data modelling; Hazard rate; Infinite divisibility; Limiting behaviour

1. Introduction

Data modeling has drawn significant interest among researchers across various disciplines such as insurance, economics, social sciences and medical sciences. There are numerous approaches to modeling and representing data, with one common method being statistical modeling of real data sets. This type of modeling plays a significant role in real-world sciences, as new applications and phenomena continuously emerge over time, increasing the demand for novel distributions. Many modern phenomena require advanced modeling techniques, but unfortunately traditional distributions could not model them. As a result, researchers sometimes introduce additional parameters to address these limitations. For example see, [1] and [5].

Recently, various statistical methods have been observed for generating these discrete distributions, including discretization of continuous distributions. These models are developed to capture diverse data having characteristics such as heavy

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tails, skewness, and varying hazard rates. Some of the discrete distributions developed in this manner are discrete Weibull distribution ([18]), discrete Rayleigh distribution ([22]), discrete Burr and discrete Pareto distribution ([17]), discrete Lindley distribution ([9]), discrete gamma distribution ([4]), discrete Chen distribution ([19]), discrete Weibull-geometric distribution ([11]), discrete Burr-Hatke distribution ([7]), discrete new generalized Pareto distribution([12]) and discrete exponential intervened Poisson distribution ([13]). These discrete models differ in their tail behaviour, parameter complexity, and suitability for specific applications, providing a wide range of models suited to various discrete data characteristics.

In this paper, a new discrete distribution is introduced and studied. For a random variable N taking non negative integer $N_0 = \{0, 1, 2, ...\}$ values, the function

$$F(n) = P(N \leq n) = 1 - [1 + \frac{a^b}{\beta}(n+1)^b]^{-\alpha} \tag{1}$$

is a proper cumulative distribution function (CDF), having four parameters, $a > 0, b > 0, \alpha > 0$ and $\beta > 0$, and also a and β are the scale parameters whereas b and α are shape parameters. since it is easily to prove that it is non-negative and strictly increasing and goes to one when $n \to \infty$.

This new distribution can be considered as an alternative to discrete Burr, discrete Pareto and discrete generalized Pareto distributions, among others. The new distribution is not only overdispersed and zero inflated but also infinitely divisible. Also, the new distribution is unimodal with a zero vertex and has a compound Poisson representation.

The main motivations for developing this lifetime model when compared to prior variants (eg: discrete Pareto, discrete Burr, discrete Lomax etc.) are:

(i) to generalizes multiple existing discrete Pareto-type distributions as special cases, thus serving as a unifying family rather than a single-purpose model.
(ii) to propose a new generalized version of the discrete Pareto with an additional shape parameter that allows independent control over the tail heaviness and decay rate, enabling better adaptation to datasets exhibiting deviations from existing Pareto-type behaviour.
(iii) to model count data with heavy tails
(iv) to consistently outperform other existing discrete Pareto distributions, as well as other widely recognized discrete distributions mentioned in the literature, in terms of fitting real data sets.
(v) to provides new closed-form expressions for key properties (probability mass function (PMF), hazard function) that are not directly derivable for some existing Lomax-based discrete models.

The rest of the paper is organized as follows: In Section 2, a new discrete distribution is introduced and some distributional properties such as limiting behaviour, infinite divisibility and bounds to moments are discussed. The reliability properties namely, hazard rates, stochastic orderings, aging intensities, stability with minima and relation with geometric distribution are also investigated. Some characterizations are obtained. In Section 3, maximum likelihood estimation method is used to estimate unknown parameters and in Section 4, a simulation study is carried out. The flexibility of the suggested distribution in modeling actual data sets is illustrated in Section 5 with a numerical example from the medical area. In Section 6, conclusions are given.

2. Proposed distribution and some properties

A random variable N taking non-negative integers 0, 1, 2,. . . with CDF as in (1) , then the PMF is given by

$$f(n) = P(N = n) = [1 + \frac{a^b}{\beta} n^b]^{-\alpha} - [1 + \frac{a^b}{\beta}(n+1)^b]^{-\alpha} \tag{2}$$

Note that (2) becomes the pmf of Discrete Pareto when $a = b = \beta = 1$. Henceforth the random variable N having pmf (2) is called by generalized discrete Pareto distribution and is denoted by $GDP(a, b, \alpha, \beta)$.
The derivation of GDP distribution is explained as follows:
The probability density function (PDF) and survival function (SF) of Lomax generator ([6]) are

$$f(x) = \theta\mu^\theta \frac{g(x)}{(1 - G(x))\{\mu - \log(1 - G(x))\}^{\theta+1}}; x \geqslant 0 \tag{3}$$

and

$$S(x) = \left[\frac{\mu}{\mu - \log(1 - G(x))}\right]^\theta; x > 0, \theta, \mu > 0 \tag{4}$$

respectively, where $G(x)$ is the cdf of the baseline model.
The Lomax generator family of distributions can generalize all classical continuous distributions. The density funtion of Lomax generator allows increased tail flexibility and has broad applications in many areas of engineering and biology. Also moments of this family plays crucial role in measuring inequality, such as, income quantiles and Lorenz and Bonferroni curves, which rely on the incomplete moments of the distribution.

Here we consider the survival discretization (SD) method. Let X be a continuous rv. Then the discerte analogue Y of X can be derived by using the sf as follows: Let $S(.)$ be the SF of the rv X. According to the SD approach, the PMF can be expressed as

$$P(Y = y) = S(y) - S(y + 1); \quad y = 0, 1, 2, 3, \ldots. \tag{5}$$

Employing SD approach on (3), the PMF of discrete Lomax generator family can be obtained as,

$$f_Y(y) = \left[\frac{\mu}{\mu - \log(1 - G(y))}\right]^\theta - \left[\frac{\mu}{\mu - \log(1 - G(y + 1)))}\right]^\theta; \quad y \in \mathbf{N_0} \tag{6}$$

where $\mu, \theta \in (0, \infty)$ and $\mathbf{N_0} = \{0, 1, 2, \ldots\}$.
By putting CDF of Weibull distribution, $G(y) = 1 - e^{-(\delta y)^{\alpha}};\ y > 0$, in (6), we obtain

$$\left[\frac{\mu}{\mu - \log(1-(1-e^{(-\delta y)^{\alpha}}))}\right]^{\theta} - \left[\frac{\mu}{\mu - \log(1-(1-e^{(-\delta(y+1))^{\alpha}}))}\right]^{\theta} = \left[\frac{\mu}{\mu + (\delta y)^{\alpha}}\right]^{\theta} - \left[\frac{\mu}{\mu + (\delta(y+1))^{\alpha}}\right]^{\theta}$$
$$= \left[1 + \frac{(\delta y)^{\alpha}}{\mu}\right]^{-\theta} - \left[1 + \frac{(\delta(y+1))^{\alpha}}{\mu}\right]^{-\theta}.$$

This is same as the PMF of GDP distribution.

Special Cases:

1. $a = \beta = 1$, GDP reduces to Discrete Burr distribution([17]).
2. $a = b = \beta = 1$, GDP reduces to Discrete Pareto distribution([17]).
3. $b = 1$, GDP reduces to Alternate discrete Pareto distribution([14])
4. $b = \beta = 1$, GDP reduces to Discrete generalized Pareto distribution ([21]).

The PMF $f(n)$ can also be written in the following form

$$f(n) = \mathrm{v}^{\log[1+\frac{a^b}{\beta} n^b]} - \mathrm{v}^{\log[1+\frac{a^b}{\beta}(n+1)^b]} \tag{7}$$

where $0 < \mathrm{v} < 1$ and $\mathrm{v} = exp(-\alpha)$.
The SF of GDP distribution is given by,

$$S(n) = \mathrm{v}^{\log[1+\frac{a^b}{\beta}(n+1)^b]} \tag{8}$$

Asymptotic behaviour

The limiting behaviour of GDP distribution corresponding to various parameter choices at the boundary are given below.

- $lim_{n \to \infty} f(n) = 0$
- $lim_{a \to 0, \infty} f(n) = 0$
- $lim_{b \to 0, \infty} f(n) = 0$
- $lim_{\beta \to 0} f(n) = 0, \quad lim_{\beta \to \infty} f(n) = \infty$
- $lim_{n \to \infty} \frac{F(t+n)}{1-F(n)} = 1$, which implies GDP is a long tailed distribution.
- $lim_{n \to \infty} \frac{P(N>n)}{n^{-\alpha}} < \infty$, when $b = 1$, which implies GDP is heavy tailed distribution.

Theorem 2.1. *GDP has logconvex PMF.*

Proof. With PMF $f(n)$, a distribution has log convexity if (see ([10]), ([2]), ([23]))

$$f^2(n+1) < f(n).f(n+2), n \in \mathbf{N_0}. \tag{9}$$

The expression corresponding to (9) is given by

$$\begin{aligned} f^2(n+1) - f(n).f(n+2) = & \mathrm{v}^{2\log(1+\frac{a^b}{\beta}(n+1)^b)} + \mathrm{v}^{2\log(1+\frac{a^b}{\beta}(n+2)^b)} - \\ & \mathrm{v}^{\log(1+\frac{a}{\beta^b}(n+1)^b)+\log(1+\frac{a}{\beta}(n+2)^b)} - \mathrm{v}^{\log(1+\frac{a^b}{\beta}n^b)+\log(1+\frac{a^b}{\beta}(n+2)^b)} + \\ & \mathrm{v}^{\log(1+\frac{a^b}{\beta}n^b)+\log(1+\frac{a^b}{\beta}(n+3)^b)} - \mathrm{v}^{\log(1+\frac{a^b}{\beta}n)+\log(1+\frac{a^b}{\beta}(n+3)^b)} \end{aligned} \tag{10}$$

The equation in (10) can be negative for $a > 0, b > 0, \beta > 0$ and $0 < \mathrm{v} < 1$, for all $n \in \mathbf{N_0}$ and also $f(1) \neq 0, f(0) \neq 0$. According to [24], a discrete model with PMF $f(n)$ is log convex if $f^2(n) < f(n-1)f(n+1)$. That is, GDP satifies $f^2(n+1) < f(n).f(n+2)$. Thus GDP distribution is log convex and as a direct consequence, GDP distribution is infinitely divisible, see [24] for more details.

□

The maximum limit for the variance in case of this infinitely divisible distribution can expressed as

$$V(n) \leq \frac{f(1)}{f(0)} = \frac{\mathrm{v}^{\log(1+\frac{a^b}{\beta})} - \mathrm{v}^{\log(1+\frac{2a^b}{\beta})}}{1 - \mathrm{v}^{\log(1+\frac{a^b}{\beta})}}.$$

Define $\eta(t) = \frac{f(t)-f(t+1)}{f(t)} = 1 - \frac{f(t+1)}{f(t)}$.

$\Delta\eta(t) = \eta(t+1) - \eta(t) = \frac{f(t+1)}{f(t)} - \frac{f(t+2)}{f(t+1)}$.
From above discussion, it is clear that log convexity is equivalent to $\Delta\eta(t) < 0$. Thus hazard rate, h(t) is non-increasing(decreasing failure rate (DFR)).

The four parameters in the proposed model each have distinct effects on the shape, skewness, and tail behaviour of the distribution:

- $\mathrm{v}(0 < \mathrm{v} < 1)$:- scale paramter, it controlls overall survival decay. As the value of v increases, decay will be slow, but increases skewness and produce heavier tails.

- $a(a > 0)$:- Large values of "a" raises early probabilities and hazard and reduces skewness by concentrating mass at small n. Small values of "a" (with large "b" values) can allow more mass at tail.

- $b(b > 0)$:- shape parameter, for smaller values of "b"tends to gradual decay and increases skewness and produce heavier tails.

- $\beta(\beta > 0)$:- determines tail thickness, small values of β produces very heavy tails and higher skewness.

The PMF plots of GDP for selected values of a, b,v, β are presented in Figure 1.

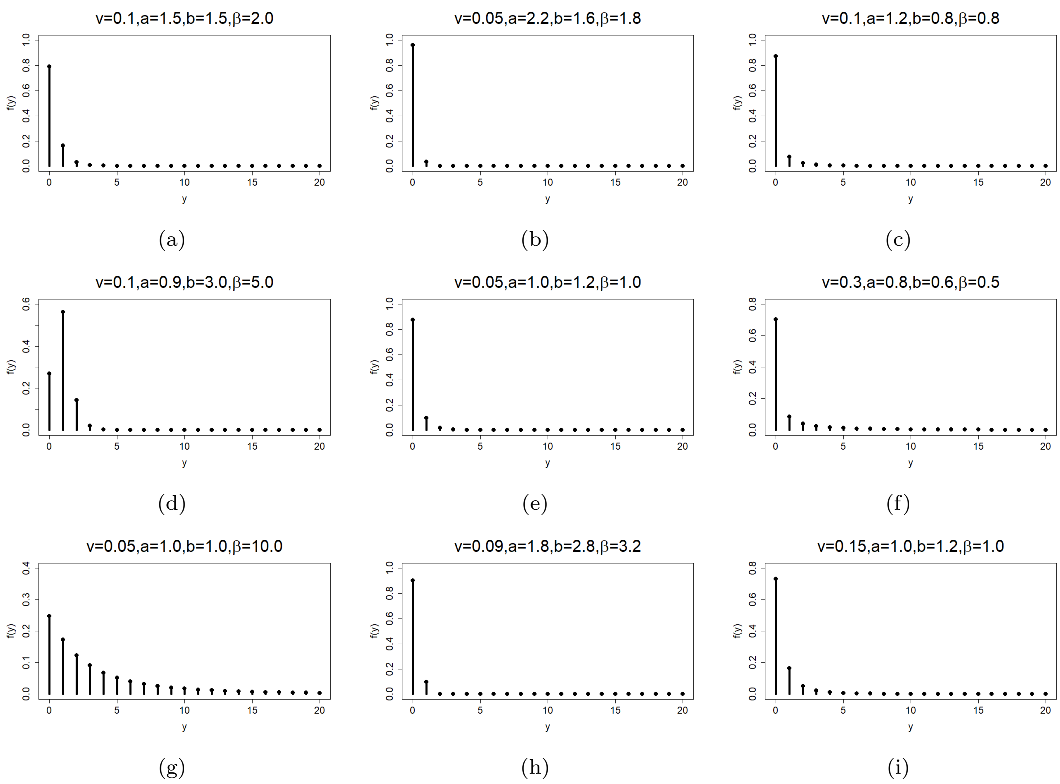


Figure 1. PMF plots of GDP for different values of v, a, b and β

Moments properties

If N follows GDP distribuion, the r^{th} moment is given by

$$\begin{aligned}
E\left(N^r\right) &= \sum_{n=0}^{\infty} n^r f(n) \\
&= \sum_{n=1}^{\infty} \left(n^r - (n-1)^r\right) S(n) \\
&\leq r \sum_{n=1}^{\infty} n^{r-1} \mathrm{v}^{\log\left(1+\left(\frac{((n+1)a)^b}{\beta}\right)\right)} \\
&\leq r \sum_{n=1}^{\infty} n^{r-1} \frac{1}{\beta} \left(a(n+1)\right)^{-\alpha b} \\
&\leq r \sum_{n=1}^{\infty} n^{r-1} a^{-\alpha b} \left(\frac{1}{\frac{1}{\beta} n^{\alpha b - r + 1}} \right).
\end{aligned}$$

Then $E(N^r)$ will be convergent if $\alpha b > r$, that is, $exp(-r/b) > \mathrm{v}$ where $\mathrm{v} = exp(-\alpha)$. That is $E(N^r)$ is finite if $\alpha b > r$.

Also, the mean lifetime of GDP can be obtained as

$$\begin{aligned}\mu(n) &= \sum_{n=1}^{\infty} \mathrm{v}^{\log(1+\frac{a^b}{\beta}n^b)} \\ &= \sum_{n=1}^{\infty} \frac{1}{(1+\frac{a^b}{\beta}n^b)^{\alpha}}\end{aligned}$$

where $\mathrm{v} = e^{-\alpha}$. This series is convergent if $\alpha b > 1$. Consequently, we have the following result.

Theorem 2.2. *$E(N^r)$ exist if and only if $exp(-r/b) > v$.*

Proof. It is easy to prove from the previous discussion. □

Probability generating function and factorial moments

The probability generating function for GDP is

$$\begin{aligned}G(s) = E(s^N) &= \sum_{n=0}^{\infty} s^n f(n); s \in (0,1) \\ &= \sum_{n=0}^{\infty} \{(1+\frac{a^b}{\beta}n^b)^{-\alpha} - (1+\frac{a^b}{\beta}(n+1)^b)^{-\alpha}\} s^n.\end{aligned} \tag{11}$$

The connection between the probabilities and the corrsponding factorial moments (see [15], page 59), is given by

$$f(n) = \sum_{r\geq 0} (-1)^r \frac{\mu\prime_{[n+r]}}{n!r!}.$$

From Table 1, it is noted that the GDP model can be used as a probability tool for modeling positively skewed data with leptokurtic shape. Moreover, it can be used to model over dispersion phenomena.

Reliability Properties

Hazard rates

The hazard rate of GDP distribution is given by

$$h(n) = \mathrm{v}^{\log\left[\frac{1+\frac{a^b}{\beta}n^b}{1+\frac{a^b}{\beta}(n+1)^b}\right]} - 1. \tag{12}$$

The hazard rate shows that $h(1) = h(0)$ when $a = (2^b - 2)^{1/b}$ or $a^b/\beta = 0$. But $a^b/\beta = 0$ cannot be happen, since $a > 0, b > 0, \beta > 0$. Then $h(n) < h(n-1)$ for all choices of parameters except for $a = (2^b - 2)^{1/b}$. Thus GDP has decreasing hazard

Table 1. Mean, Variance, Skewness and Kurtosis for various values of a, b, v and β

Parameter	Mean	Variance	Skewness	Kurtosis
a=0.1 b=0.1 v=0.9 $\beta = 0.1$	0.7261	36.5928	118.7194	135.0431
a=1.0 b=0.1 v=0.9 $\beta = 0.1$	0.7219	36.3543	119.4594	135.8936
a=5.0 b=0.1 v=0.9 $\beta = 0.1$	0.7174	36.1147	120.2410	136.7833
a=5.0 b=0.1 v=0.9 $\beta = 0.5$	0.6805	34.5757	126.5873	143.6903
a=5.0 b=0.1 v=0.9 $\beta = 2.0$	0.4434	22.9707	194.9475	219.7919

rate.
The reverse hazard rate is given by

$$h^*(n) = \frac{\mathrm{v}^{\log[1+\frac{a^b}{\beta}y^b]} - \mathrm{v}^{\log[1+\frac{a^b}{\beta}(y+1)^b]}}{1 - \mathrm{v}^{\log[1+\frac{a^b}{\beta}(y+1)^b]}}. \tag{13}$$

Since GDP has decreasing hazard rate , the reverse hazard rate is also decreasing.

The cumulative hazard rate function is given by

$$\begin{aligned} H_1(n) &= -\log S(n) \\ &= -\log[\mathrm{v}^{\log[1+\frac{a^b}{\beta}(n+1)^b]}] \\ &= \log[1 + \frac{a^b}{\beta}(n+1)^b]^{\alpha}. \end{aligned} \tag{14}$$

The alternative hazard rate function is given by

$$\begin{aligned} h_1(n) &= -\log[\frac{S(n)}{S(n+1)}] \\ &= \alpha \log\left[\frac{1 + \frac{a^b}{\beta}(n+1)^b}{1 + \frac{a^b}{\beta}(n+2)^b}\right]. \end{aligned} \tag{15}$$

Corollary 2.3. *The alternative hazard rate, $h_1(n)$ presented in (15), uniquely determines GDP distribution.*

The alternative reverse hazard rate is given by,

$$\begin{aligned} h_1^* &= \log[\frac{F(n)}{F(n-1)}] \\ &= \log[\frac{1 - (1 + \frac{a^b}{\beta}(n+1)^b)^-\alpha}{1 - (1 + \frac{a^b}{\beta}(n+2)^b)^-\alpha}]. \end{aligned} \tag{16}$$

Stocashastic Orderings

Here we apply stochastic ordering as a tool for assessing the comparative behaviour of the random variables. The orders under consideration here are the stochastic order $\leq_{st}$, the expectation order $\leq_E$ and the hazard rate order $\leq_h$, which are crucial in reliability studies because they provide a rigorous framework to compare lifetimes of components or systems. These orders help determine which system is more reliable or fails earlier on average, supporting decision-making in maintenance, risk assessment, and design optimization.

Theorem 2.4.
- *Let $N_1 \sim GDP(a_1, b, v, \beta)$ and $N_2 \sim GDP(a_2, b, v, \beta)$. If $a_1 > a_2$, then $N_1 \leq_{st} N_2$.*

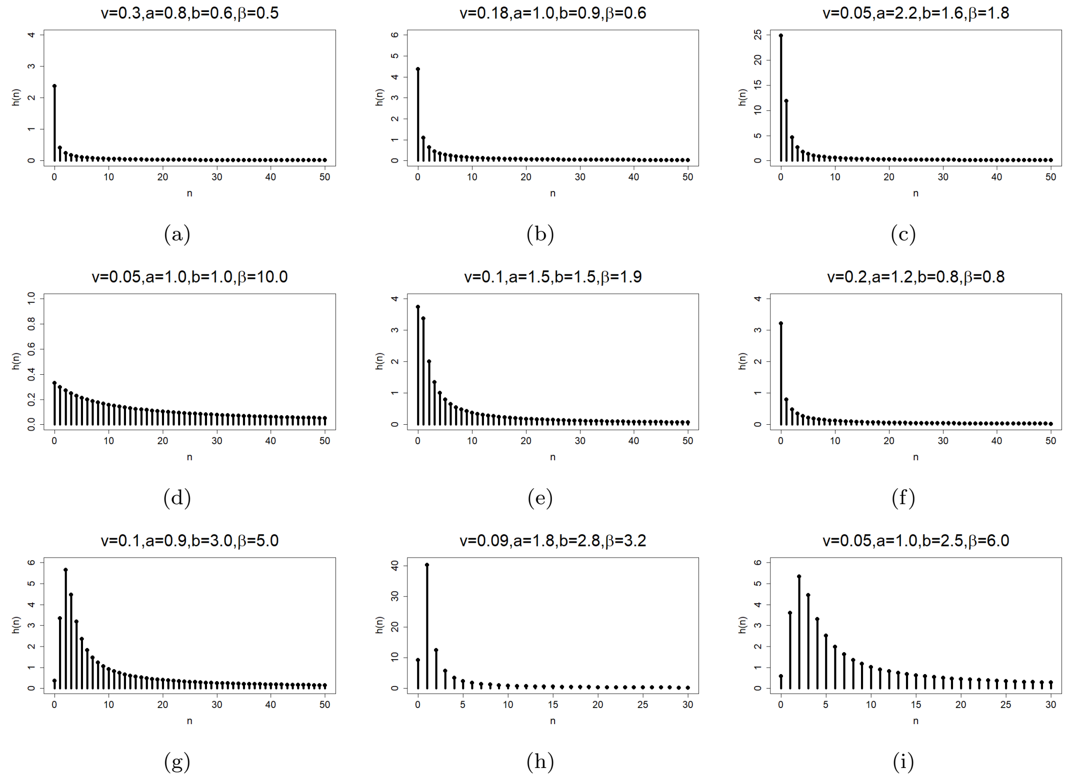


Figure 2. Failure rates plots of GDP(v,a,b,β) for different values of v,a,b and β

- *Let* $N_1 \sim GDP(a, b_1, v, \beta)$ *and* $N_2 \sim GDP(a, b_2, v, \beta)$. *If* $b_1 > b_2$, *then* $N_1 \leq_{st} N_2$.
- *Let* $N_1 \sim GDP(a, b, v_1, \beta)$ *and* $N_2 \sim GDP(a, b, v_2, \beta)$. *If* $v_1 > v_2$, *then* $N_1 \geq_{st} N_2$.
- *Let* $N_1 \sim GDP(a, b, v, \beta_1)$ *and* $N_2 \sim GDP(a, b, v, \beta_2)$. *If* $\beta_1 > \beta_2$, *then* $N_1 \geq_{st} N_2$.

Corollary 2.5. *From Theorem 3, the expectation and hazard rate ordering are as follows.*

- *Suppose* $N_1 \sim GDP(a_1, b, v, \beta)$ *and* $N_2 \sim GDP(a_2, b, v, \beta)$. *If* $a_1 > a_2$, *then* $N_1 \leq_E N_2$ *and* $N_1 \leq_h N_2$.
- *Suppose* $N_1 \sim GDP(a, b_1, v, \beta)$ *and* $N_2 \sim GDP(a, b_2, v, \beta)$. *If* $b_1 > b_2$, *then* $N_1 \leq_{st} N_2$ *and* $N_1 \leq_h N_2$.
- *Suppose* $N_1 \sim GDP(a, b, v_1, \beta)$ *and* $N_2 \sim GDP(a, b, v_2, \beta)$. *If* $v_1 > v_2$, *then* $N_1 \geq_{st} N_2$ *and* $Y_1 \geq_h N_2$.
- *Suppose* $N_1 \sim GDP(a, b, v, \beta_1)$ *and* $N_2 \sim GDP(a, b, v, \beta_2)$. *If* $\beta_1 > \beta_2$, *then* $N_1 \geq_{st} N_2$ *and* $N_1 \geq_h N_2$.

Discrete aging intensity and Discrete alternative aging intensity

The rate of instantaneous hazard rate to hazard rate average is the aging intensity L. The discrete aging intensity and discrete alternative aging intensity of GDP distribution are given, respectively, by

$$L(n) = y\left[1 - \frac{\log\left(1-(1+\frac{a^b}{\beta}n^b)^{-\alpha}\right)}{\log\left(1-(1+\frac{a^b}{\beta}(n+1)^b)^{-\alpha}\right)}\right] \tag{17}$$

and

$$L^*(n) = \frac{\log\left(\log(1+\frac{a^b}{\beta}n^b)\right) - \log\left(\log(1+\frac{a^b}{\beta}(n-1)^b)\right)}{\log n - \log(n-1)}. \tag{18}$$

Corollary 2.6. *If for a discrete random variable N, discrete aging intensity L has the form in (12) for* $n = 1, 2, 3, ..\& a > 0, b > 0, 0 < v < 1, \beta > 0$*, then N follows GDP distribution.*

The odds ratios are given by

$$\begin{aligned} w(n) &= \frac{1}{F(n)} - 1 \\ &= \frac{\mathrm{v}^{\log\left[1+\frac{a^b}{\beta}(n+1)^b\right]}}{1-\mathrm{v}^{\log\left[1+\frac{a^b}{\beta}(n+1)^b\right]}} \end{aligned} \tag{19}$$

and,

$$\begin{aligned} \bar{w}(n) &= \frac{1}{S(n)} - 1 \\ &= \mathrm{v}^{-\log\left[1+\frac{a^b}{\beta}(n+1)^b\right]} - 1. \end{aligned} \tag{20}$$

Corollary 2.7. *Both* $w(n)$ *and* $\bar{w}(n)$ *presented in (19) and (20) uniquely determines the distribution.*

Characterizatons

In order to comprehend the patterns exhibited by data generated through a specific process, it is essential to articulate its behavior through an appropriate probability distribution. Hence, characterizing a distribution becomes a crucial challenge in applied sciences, as investigators are keen to ascertain whether their model adheres to the correct distribution. In pursuit of this goal, investigators depend on conditions that determine whether their model aligns with the chosen distribution. Here we obtain three characterizations of GDP distribution based on: a) the conditional expectation of certain function of the random variable; b) the hazard rate function and c) reverse hazard rate function. The conditional expectation approach helps in moment-type properties and can be useful for parameter estimation. The characterization based on hazard rate function focuses on the failure-rate behaviour, making it relevant in reliability and survival analysis while characterization based on reverse hazard rate obatins the distribution's behaviour from the lower tail, which is important in modelling early-occurring events.

Based on conditional expectation of certain function of the random variable

Proposition 2.8. *$N \sim GDP(a,b,v,\beta)$ if and only if*

$$E\left\{v^{\log(1+\frac{a^b}{\beta}n^b)} + v^{\log(1+\frac{a^b}{\beta}(n+1)^b)}|N>k\right\} = v^{\log(1+\frac{a^b}{\beta}(k+1)^b)}. \tag{21}$$

Proof. If N has PMF given in (2), then LHS of (21) will be

$$\begin{aligned}
&[1-F(k)]^{-1}\textstyle\sum_{n=k+1}^{\infty}\left[v^{\log\left[1+\frac{a^b}{\beta}n^b\right]} + v^{\log\left[1+\frac{a^b}{\beta}(n+1)^b\right]}\right]\times\\
&\left[v^{\log\left[1+\frac{a^b}{\beta}n^b\right]} - v^{\log\left[1+\frac{a^b}{\beta}(n+1)^b\right]}\right]\\
&=[1-F(k)]^{-1}\textstyle\sum_{n=k+1}^{\infty} v^{2\log\left[1+\frac{a^b}{\beta}n^b\right]} - v^{2\log\left[1+\frac{a^b}{\beta}(n+1)^b\right]}\ .\\
&=\left[v^{2\log\left[1+\frac{a^b}{\beta}(k+1)^b\right]}\right]\left[v^{-\log\left[1+\frac{a^b}{\beta}(k+1)^b\right]}\right]\\
&=v^{\log\left[1+\frac{a^b}{\beta}(k+1)^b\right]}
\end{aligned}$$

Conversely, if (21) holds, then

$$\begin{aligned}
&\textstyle\sum_{n=k+1}^{\infty}\left[v^{\log\left[1+\frac{a^b}{\beta}n^b\right]} + v^{\log\left[1+\frac{a^b}{\beta}(n+1)^b\right]}\right]f(n)\\
&=\textstyle\sum_{n=k+1}^{\infty} v^{2\log\left[1+\frac{a^b}{\beta}n^b\right]} - v^{2\log\left[1+\frac{a^b}{\beta}(n+1)^b\right]}\\
&=v^{2\log\left[1+\frac{a^b}{\beta}(k+1)^b\right]}\\
&=(1-F(k))v^{\log\left[1+\frac{a^b}{\beta}(k+1)^b\right]}\\
&=(1-F(k+1)+f(k+1))v^{\log\left[1+\frac{a^b}{\beta}(k+1)^b\right]}.
\end{aligned} \tag{22}$$

Also we have,

$$\begin{aligned}
&\textstyle\sum_{n=k+2}^{\infty}\left[v^{\log\left[1+\frac{a^b}{\beta}n^b\right]} + v^{\log\left[1+\frac{a^b}{\beta}(n+1)^b\right]}\right]f(y)\\
&=(1-F(k+1))v^{\log\left[1+\frac{a^b}{\beta}(k+2)^b\right]}.
\end{aligned} \tag{23}$$

Now, substracting (23) from (22), we arrive at

$$\begin{aligned}
&v^{\log\left[1+\frac{a^b}{\beta}(k+2)^b\right]}f(k+1)=\\
&(1-F(k+1))\left[v^{\log\left[1+\frac{a^b}{\beta}(k+1)^b\right]} - v^{\log\left[1+\frac{a^b}{\beta}(k+2)^b\right]}\right]\\
&\text{Hence,}\quad h(k)=\tfrac{f(k+1)}{1-F(k+1)}=\frac{v^{\log\left[1+\frac{a^b}{\beta}(k+1)^b\right]}-v^{\log\left[1+\frac{a^b}{\beta}(k+2)^b\right]}}{v^{\log\left[1+\frac{a^b}{\beta}(k+2)^b\right]}}
\end{aligned}$$

which, in view of (12), implies that N has PMF in (2). $\square$

Based on hazard rate function

Proposition 2.9. *$N \sim$ GDP(a, b, v, β) if and only if its hazard rate function satisfies the difference equation*

$$h(k+1) - h(k) = v^{\log\left[\frac{1+\frac{a^b}{\beta}(k+2)^b}{1+\frac{a^b}{\beta}(k+1)^b}\right]} - v^{\log\left[\frac{1+\frac{a^b}{\beta}(k+1)^b}{1+\frac{a^b}{\beta}k^b}\right]}; \quad k \in \mathbf{N_0} \tag{24}$$

with boundary condition

$$h(0) = v^{\log[1+a^b/\beta]} - 1.$$

Proof. If N has PMF in (2) then clearly (24) holds. Now if (24) holds, then for every $N \in \mathbf{N_0}$, we have

$$\sum_{k=0}^{N-1} h(k+1) - h(k) = \sum_{k=0}^{N-1} v^{\log\left[\frac{1+\frac{a^b}{\beta}(k+2)^b}{1+\frac{a^b}{\beta}(k+1)^b}\right]} - v^{\log\left[\frac{1+\frac{a^b}{\beta}(k+1)^b}{1+\frac{a^b}{\beta}k^b}\right]}$$

$$h(n) - h(0) = v^{\log[1+a^b/\beta]} - v^{\log\left[\frac{1+\frac{a^b}{\beta}(n+1)}{1+\frac{a^b}{\beta}n^b}\right]}.$$

In view of the fact $h(0) = v^{\log[1+a^b/\beta]} - 1$, from the last equation we have

$$h(n) = v^{\log\left[\frac{1+\frac{a^b}{\beta}(n+1)^b}{1+\frac{a^b}{\beta}n^b}\right]} - 1$$

which in view of (12), implies N has PMF in (2). $\square$

Based on reverse hazard rate function

Proposition 2.10. *$N \sim$ GDP(a, b, v, β) if and only if its reverse hazard rate function satisfies the difference equation*

$$h^*(k+1) - h^*(k) = \frac{v^{\log(1+\frac{a^b}{\beta}(k+1)^b} - v^{\log(1+\frac{a^b}{\beta}(k+2)^b)}}{1 - v^{\log(1+\frac{a^b}{\beta}(k+2)^b)}} - \frac{v^{\log(1+\frac{a^b}{\beta}k^b} - v^{\log(1+\frac{a^b}{\beta}(k+1)^b)}}{1 - v^{\log(1+\frac{a^b}{\beta}(k+1)^b)}}; \quad k \in \mathbf{N_0} \tag{25}$$

with boundary condition $h^*(0) = 1$.

Proof. If N has PMF in (2) then clearly (25) holds. Now if (25) holds, then for every $N \in \mathbf{N_0}$, we have

$$\sum_{k=0}^{n-1} h^*(k+1) - h^*(k) = \sum_{k=0}^{n-1} \frac{v^{\log(1+\frac{a^b}{\beta}(k+1)^b} - v^{\log(1+\frac{a^b}{\beta}(k+2)^b)}}{1 - v^{\log(1+\frac{a^b}{\beta}(k+2)^b)}} - \frac{v^{\log(1+\frac{a^b}{\beta}k^b} - v^{\log(1+\frac{a^b}{\beta}(k+1)^b)}}{1 - v^{\log(1+\frac{a^b}{\beta}(k+1)^b)}}.$$

Or,

$$h^*(n) - h^*(0) = \frac{v^{\log(1+\frac{a^b}{\beta}n^b)} - v^{\log(1+\frac{a^b}{\beta}(n+1)^b)}}{1 - v^{\log(1+\frac{a^b}{\beta}(n+1)^b)}} - 1.$$

In view of the fact that $h^*(0) = 1$, from the last equation we have

$$h^*(n) = \frac{v^{\log(1+\frac{a^b}{\beta}n^b} - v^{\log(1+\frac{a^b}{\beta}(n+1)^b)}}{1 - v^{\log(1+\frac{a^b}{\beta}(n+1)^b)}}$$

which in view of (13), implies N has PMF in (2). □

An alternative parametrization

To study the extreme cases of the parameters, when a, α and β become either 0 or infinity, it is useful for introduce an alternative parametrization of the GDP distribution. Here we take $b = 1$. The new parameters are defined as

$$\lambda = \frac{1}{\alpha}$$

and

$$\theta = 1 - e^{-\frac{a}{\lambda\beta}}$$

and the PMF of the GDP takes on the form

$$f(n) = \left(\frac{1}{1 - \lambda n log(1-\theta)}\right)^{1/\lambda} - \left(\frac{1}{1 - \lambda(n+1)log(1-\theta)}\right)^{1/\lambda}; n \in \mathbf{N_0} \qquad (26)$$

We shall use the notation GDP(λ, θ) for the PMF given in (26), where $\lambda > 0$ and $0 < \theta < 1$.
This reparameterization have several advantages:

- It is easy to interpret the probability since θ lies between o and 1.
- λ controls how far the model is from the geometric case and affects the tail behaviour.
- It easier to study special cases and limits.

Proposition 2.11. *Let N follows GDP(λ, θ) distribution with $\theta \in (0, 1)$. The parameter $\alpha \to 0$, then N converges in distribution to a geometric random variable.*

Proof. We have

$$lim_{z\to\infty}(1 + \frac{x}{z})^z = e^z. \qquad (27)$$

Substitute $z = \frac{1}{\lambda}$ in (27), we can arrive at

$$f(n) = \left(\frac{1}{1 - \frac{nlog(1-\theta)}{1/\lambda}}\right)^{1/\lambda} - \left(\frac{1}{1 - \frac{(n+1)log(1-\theta)}{1/\lambda}}\right)^{1/\lambda}$$

converges to

$$\frac{1}{e^{-n\log(1-\theta)}} - \frac{1}{e^{-(n+1)\log(1-\theta)}}$$

$$\begin{aligned}
&= (1-\theta)^n - (1-\theta)^{n+1} \\
&= (1-\theta)^n \theta
\end{aligned}$$

This is the PMF of geometric distribution with parameter θ. This result shows that when $\alpha \to 0$, the GDP distribution simplifies to a geometric model. Hence the proof. □

Remark 1. The GDP model includes the geometric distribution as a special case, providing additional flexibility for modeling count data that may range from geometric-like behaviour to heavier-tailed patterns as α increases.

Lemma 2.12. *Let $k \in \mathbf{N_0}$ and $0 < \theta < 1$. Then $lim_{\lambda\to\infty}\frac{log(1-\lambda klog(1-\theta))}{\lambda} = 0$.*

Proof. When $\lambda \to \infty$, both numerator and denominator converges to ∞. Applying L'hospital rule, we get

$$\frac{\frac{\partial}{\partial\lambda}log(1-\lambda klog(1-\theta))}{\frac{\partial}{\partial\lambda}\lambda} = \frac{-klog(1-\theta)}{1-\lambda klog(1-\theta)}$$

which converges to zero. This is the proof. □

Proposition 2.13. *Suppose that Y has a gamma distribution with shape parameter $1/\lambda$ and scale parameter $\gamma = \frac{-1}{\lambda log(1-\theta)}$ so that the PDF of Y is $f(y) = \frac{\gamma^{1/\lambda}}{\Gamma(1/\lambda)}y^{1/\lambda-1}e^{-\gamma y}, y > 0$, and the conditional distribution of $T|Y = y$ has a geometric distribution with parameter $q = 1 - e^{-y}$. Then $T \sim GDP(\lambda, \theta)$.*

Proof. Let $f_{\gamma,\lambda}(y)$ be the density of the random variable Y. Then the PMF of T can be obtained as

$$\begin{aligned}
P(T = n) &= \int_0^\infty P(T = n|Y = y)f(y)\,dy \\
&= \int_0^\infty (e^{-ny} - e^{-(n+1)y})\frac{\gamma^{1/\lambda}}{\Gamma(1/\lambda)}y^{1/\lambda-1}e^{-\gamma y}\,dy \\
&= \left(\frac{1}{1-\lambda nlog(1-\theta)}\right)^{1/\lambda} - \left(\frac{1}{1-\lambda(n+1)log(1-\theta)}\right)^{1/\lambda}
\end{aligned}$$

Hence the proof. □

Some theorems related to GDP distribution

Theorem 2.14. *Let* $Z = \min_{i \leq i \leq m} N_i$, *and* $N_i's$ $(i = 1, 2, 3, \ldots m)$ *be non-negative independent and identically distributed (iid) integer valued r.v. Then* $Z \sim GDP(a, b, v^m, \beta)$ *iff* $N_i \sim GDP(a, b, v, \beta)$.

Proof. Let $N_i(i = 1, 2, 3, \ldots m)$ be iid GDP(a, b, v, β). Then $S(n) = v^{\log(1+\frac{a^b}{\beta}(n+1)^b)}; N = 0, 1, 2, 3, \ldots$

Consider

$$\begin{aligned} S(z) = P(Z \geqslant z) &= [P(N_1 \geqslant z)]^m \\ &= (v^m)^{\log(1+\frac{a^b}{\beta}(z+1)^b)} \end{aligned}$$

Thus, $Z \sim$ GDP (a, b, v^m, β) for all $z = 0, 1, 2, 3, \ldots$
Conversely, let $S(z) = v^m \log\left(1 + \frac{a^b}{\beta}(z+1)^b\right); z = 0, 1, 2, 3 \ldots$
We know that,

$$\begin{aligned} S(n) = P[N_1 \geqslant n] =& [P[Z \geqslant n]]^{1/m} \\ =& v^{\log(1+\frac{a^b}{\beta}(n+1)^b)}; n = 0, 1, 2, 3, \ldots \end{aligned}$$

Hence the theorem. □

Remark 2. The property established through the above property has application in reliability modelling of a series system. That is, the series system with identical components having the GDP distribution.

Theorem 2.15. *Let* $N_i's$ $(i = 1, 2, 3, \ldots m)$ *be non-negative independently distributed integer valued r.v, and* $Z = \min_{i \leq i \leq m} N_i$. *Then Z is GDP* (a, b, v, β) *if* $N_i's$ *are GDP* $(a, b, v_i, \beta,)$ *where* $v = \prod_{i=1}^m v_i$.

Proof. Follows easily. □

Theorem 2.16. *If N is a non negative integer valued r.v and 's' is a positive number. Then* $N_s = N^s \sim GDP(a, b/s, v, \beta)$ *if* $N_i \sim GDP(a, b, v, \beta)$.

Proof. Let $N \sim GDP(a, b, v, \beta)$ for all $n = 0, 1, 2, 3 \ldots$.

$$\begin{aligned} P(N_s \geq n) &= P(N^s \geq n) \\ &= P(N \geq n^{1/s}) \\ &= \frac{1}{1 + \frac{a^b}{\beta} n^{b/s}} \\ &= e^{-\alpha log(1+\frac{a^b}{\beta} n^{b/s})} \\ &= v^{log(1+\frac{a^b}{\beta} n^{b/s})} \end{aligned}$$

That is, $N_s \sim GDP(a, b/s, v, \beta)$. □

Theorem 2.17. *If $N \sim \text{GDP}(a, b, v, \beta)$, then $Z = \log(\frac{a^b}{\beta} N^b + 1)$ follows geometric distribution with success probability v , where $v = e^{-\alpha}$.*

Proof.

$$\begin{aligned}
&P(Z \geqslant z) = P[\log(\tfrac{a^b}{\beta} N^b + 1) \geqslant z] \\
&= P\left[N \geqslant \left((e^z - 1)\beta/a^b\right)^{1/b}\right] \\
&= v^{\log(1 + a^b/\beta((e^z-1)\beta/a^b))} \\
&= v^z, \quad z = 0, 1, 2, 3, \ldots
\end{aligned}$$

This is the survival of Geometric r.v. Thus $Z \sim \text{Geo}(v)$. □

Theorem 2.18. *If N follows geometric distribution with success probability v, then $Z = [\frac{\beta}{a^b}(e^N - 1)]^{1/b} - 1 \sim GDP(a, b, v, \beta)$.*

Proof. Consider

$$\begin{aligned}
&p(Z \geqslant z) \\
&= P\left[\left[(\tfrac{\beta}{a^b}(e^N - 1))^{1/b} - 1\right] \geqslant z\right] \\
&= P\left[e^N - 1 \geqslant \tfrac{a^b}{\beta}(z + 1)^b\right] \\
&= P\left[e^N \geqslant 1 + \tfrac{a^b}{\beta}(z + 1)^b\right] \\
&= P[N \geqslant \log(1 + \tfrac{a^b}{\beta}(z + 1)^b)] \\
&= v^{\log(1 + \frac{a^b}{\beta}(z+1)^b)}
\end{aligned}$$

Hence the theorem. □

Theorem 2.19. *If N_1 and N_2 are independent and follow Geo(v) and GDP(a, b, α, β) and $a^b/\beta = 1$, respectively, then $Z = min(N_1, N_2) \sim NGDP(v, \alpha)$ (see [3]).*

Proof. The reliability function of $N_1 \sim$ Geo (v) and $N_2 \sim$ GDP are v^n and $(n+2)^{-\alpha}$ respectively. Hence the reliability function of $min(N_1, N_2)$ is

$$\begin{aligned}
S_Z(z) &= P(min(N_1, N_2) \geqslant z) \\
&= v^z * (z + 2)^{-\alpha} \\
&= \frac{v^z}{(z + 2)^\alpha}
\end{aligned}$$

This is the survival of NGDP(v, α). □

Theorem 2.20. *If $N \sim$ GDP (a, b, v, β), then*

$$\frac{P(N > t)}{\left(1 + \frac{a^b}{\beta}(1 + t)^b\right)^{-\alpha}} \to 1 \text{ as } t \to \infty.$$

Proof. As $t \to \infty$, here we consider $a = a(t)$ be a unique integer such that $a(t) \leq t \leq a(t) + 1$.

As a consequence,

$$S(a(t)) \geqslant P(Y > t) \geqslant S(a(t)+1)$$

Therefore,

$$\left[\frac{(1+a^b/\beta(1+t)^b)}{(1+a^b/\beta(1+a(t))^b}\right]^{\alpha} \geqslant \frac{P(Y>t)}{[1+a^b/\beta(1+t)^b]^{-\alpha}} \geqslant \left[\frac{1+a^b/\beta(1+t)^b}{1+a^b/\beta(2+a(t))^b}\right]^{\alpha}$$

The term in the middle is bounded by two terms which converges to 1 as $t \to \infty$ since $t/a(t) \to 1$. □

Quantile function and Data generation

The q^{th} quantile of the GDP distribution is given by

$$n_q = \left[\left((1-q)^{-1/\alpha} - 1\right)\frac{\beta}{a^b}\right]^{1/b} - 1. \tag{28}$$

In particular, median is given by,

$$n_{0.5} = \left[\left(2^{1/\alpha} - 1\right)\frac{\beta}{a^b}\right]^{1/b} - 1$$

The data can be generated from the proposed distribution using inverse transformation method . Let q be the random number, particularly generated from uniform distribution in the unit interval. Then using (24) random number is generated from GDP distribution.

3. Estimation

In this section, we examines method of maximum likelihood estimation as point estimation. The bias, mean squared errors (MSEs) mean relative errors (MREs) and coverage probability (CP) of the estimators are obtained from a Monte Carlo simulation experiment. Let $N_1, N_2, ..., N_m$ be a random sample from GDP distribution with the corresponding observed values $n_1, n_2, ...n_m$.

Method of maximum likelihood estimation

The log- likelihood function is given by

$$L = \sum_{i=1}^{m} \log\left[\mathrm{V}_1(n) - \mathrm{V}_2(n)\right] \tag{29}$$

where $\mathrm{V}_1(n) = \mathrm{v}^{\log(1+\frac{a^b}{\beta}n_i^b)}$ and $\mathrm{V}_2(n) = \mathrm{v}^{\log(1+\frac{a^b}{\beta}(n_i+1)^b)}$.

The maximum likelihood estimator (MLE) of a, say $\hat{a}$, is formulated as

$$\hat{a} = arg_{a\in(0,\infty)} maxL$$

Then, $\hat{a}$ is the solution of the non linear equation given below

$$\frac{\partial L}{\partial a} = \sum_{i=1}^{m} \frac{b/\beta a^{b-1} \log \mathrm{v} \left(n_i^b \mathrm{V}_1(n) \log \prime \left(1 + \frac{a^b}{\beta} n_i^b\right)\right)}{\mathrm{V}_1(n) - \mathrm{V}_2(n)} - \sum_{i=1}^{m} \frac{b/\beta a^{b-1} \log \mathrm{v} \left((n_i+1)^b \mathrm{v}^{\log\left(1+\frac{a^b}{\beta}(n_i+1)^b\right)} \log \prime \left(1 + \frac{a^b}{\beta}(n_i+1)^b\right)\right)}{\mathrm{V}_1(n) - \mathrm{V}_2(n)} = 0. \tag{30}$$

The MLE of b, say $\hat{b}$, is expressed as

$$\hat{b} = arg_{b\in(0,\infty)} maxL$$

Then, the solution of the following non linear equation is $\hat{b}$,

$$\frac{\partial L}{\partial b} = \sum_{i=1}^{m} \frac{\mathrm{V}_1(n) \log \mathrm{v} \left(\frac{(an_i)^b}{\beta} (\log a + \log n_i)\right) \log \prime \left(1 + \frac{a^b}{\beta} n_i^b\right)}{\mathrm{V}_1(n) - \mathrm{V}_2(n)} - \sum_{i=1}^{m} \frac{\mathrm{V}_2(n) \log \mathrm{v} \left(\frac{(a(n_i+1)^b}{\beta} (\log a + \log(n_i+1))\right) \log \prime \left(1 + \frac{a^b}{\beta}(n_i+1)^b\right)}{\mathrm{V}_1(n) - \mathrm{V}_2(n)} = 0 \tag{31}$$

The MLE of v, say $\hat{\mathrm{v}}$, is expressed as

$$\hat{\mathrm{v}} = arg_{\mathrm{v}\in(0,1)} maxL$$

Then, $\hat{\mathrm{v}}$ is the solution of the following non linear equation

$$\frac{\partial L}{\partial \mathrm{v}} = \sum_{i=1}^{m} \frac{\mathrm{V}_1(n)^{-1} \log \left(1 + \frac{a^b}{\beta} n_i^b\right) - \mathrm{V}_2(n)^{-1} \log \left(1 + \frac{a^b}{\beta}(n_i+1)^b\right)}{\mathrm{V}_1(n) - \mathrm{V}_2(n)} = 0 \tag{32}$$

The MLE of β, say $\hat{\beta}$, is expressed as

$$\hat{\beta} = arg_{\beta\in(0,1)} maxL$$

Then, $\hat{\beta}$ is the solution of the following non linear equation

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{m} \frac{-\frac{(an_i)^b}{\beta^2} \log \text{v} \text{V}_1(n) log\prime \left(1 + \frac{a^b}{\beta} n_i^b\right)}{\text{V}_1(n) - \text{V}_2(n)} + \sum_{i=1}^{m} \frac{-\frac{(a(n_i+1))^b}{\beta^2} \log \text{v} \text{V}_2(n) \log\prime \left(1 + \frac{a^b}{\beta} (n_i+1)^b\right)}{\text{V}_1(n) - \text{V}_2(n)} = 0 \tag{33}$$

The MLE of a, b, v, β, say $\hat{\tau}$, where $\tau = (a, b, \text{v}, \beta)$, cannot be obtained explicitly. Therefore, numerical methods, such as the Newton-Raphson are used to determine the estimates. Here we employ R programming for finding the estimates. The asymptotic variance-covariance matrix of the MLEs of parameters a, b, v and β are obtained by inverting the Fisher's information matrix with elements which are negative expected values of second order derivatives of the log-likelihood function (L). Using the general theory of MLEs, the asymptotic distribution of ($\hat{a}, \hat{b}, \hat{\text{v}}$ and $\hat{\beta}$) is a multivariate normal with mean (a,b, v, β) and variance-covariance matrix is given by

$$\begin{bmatrix} E(-\frac{\partial^2 L}{\partial a^2}) & E(-\frac{\partial^2 L}{\partial a \partial b}) & E(-\frac{\partial^2 L}{\partial a \partial \text{v}}) & E(-\frac{\partial^2 L}{\partial a \partial \beta}) \\ E(-\frac{\partial^2 L}{\partial b \partial a}) & E(-\frac{\partial^2 L}{\partial b^2}) & E(-\frac{\partial^2 L}{\partial b \partial \text{v}}) & E(-\frac{\partial^2 L}{\partial b \partial \beta}) \\ E(-\frac{\partial^2 L}{\partial \text{v} \partial a}) & E(-\frac{\partial^2 L}{\partial \text{v} \partial b}) & E(-\frac{\partial^2 L}{\partial \text{v}^2}) & E(-\frac{\partial^2 L}{\partial \text{v} \partial \beta}) \\ E(-\frac{\partial^2 L}{\partial \beta \partial a}) & E(-\frac{\partial^2 L}{\partial \beta \partial b}) & E(-\frac{\partial^2 L}{\partial \beta \partial \text{v}}) & E(-\frac{\partial^2 L}{\partial \beta^2}) \end{bmatrix}.$$

4. Simulation Experiments

In this section, a simulation study is conducted to observe the performances of the MLE, based on 1000 replications. For this purpose, we generate samples of different sizes, i.e. n = 50, 100, 250 and 500 from GDP distribution. To avoids identifiability problems fix $\beta = 1$ and estimate (a,b,v)) and the model parameters are taken to be as (a,b,v)= (0.2,1.0,0.3) and (0.1, 0.9, 0.2). The simulation is performed in the following steps.

- Step 1: Generate a random sample $N_1, N_2, ...N_m$ using the inverse transformation technique.
- Step 2: Compute the MLE.
- Step 3: Step 1 and Step 2 are repeated 2000 times.
- Step 4: The bias, MSEs, MREs and coverage probability (CP) are calculated, respectively, as

$$Bias_\tau(\hat{\tau}_i) = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\tau}_i - \tau),$$

$$MSEs_\tau(\hat{\tau}_i) = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\tau}_i - \tau)^2,$$

Parameter	Sample size(n)	Bias	MSE	MRE	CP
a=0.2	50	0.0396	0.0280	0.6137	0.607
	100	0.0368	0.0170	0.5005	0.970
	250	0.0325	0.0082	0.3585	0.976
	500	0.0295	0.0059	0.2985	0.984
b=1.0	50	0.0726	0.1138	0.2415	0.940
	100	0.0293	0.0530	0.1738	0.944
	250	0.0121	0.0176	0.1051	0.951
	500	0.0025	0.0087	0.0734	0.966
v=0.3	50	-0.0203	0.0380	0.5488	0.849
	100	-0.0177	0.0268	0.4592	0.870
	250	-0.0176	0.0124	0.2969	0.908
	500	-0.0062	0.0062	0.2088	0.928

Table 2. Simulation results for parameters a=0.2, b=1.0 and v=0.3

$$MREs_{\tau}(\hat{\tau}_i) = \frac{1}{1000}\sum_{i=1}^{1000}\frac{|\hat{\tau}_i - \tau|}{\tau},$$

where $\hat{\tau}_i$ is the estimate of τ in the i^{th} cycle. Also, we constructed 95% intervals for each parameter in every replication and calculated the coverage probability as the proportion of times the true parameter value was contained within the interval.

The simulation results are displayed in Table 2 and Table 3 for various sample sizes and parameter a, b and v. It is concluded from Table 2 and Table 3 that all estimates are asymptotically unbiased and consistent. When the sample size increases, bias, MSEs, and MREs of MLEs are approaches to zero. From Figure 3 and Figure 4, shows as the sample size increases from 50 to 500, the spread of the MLE estimates narrows considerably, and the median line converges toward the true parameter value, indicating both reduced variance and bias.

5. Real data analysis and its illustration

In this section, the GDP distribution is fitted to the number of warts observed from warts treatment results of 90 patients using immunotherapy. For more details about the data set see [16]. The mean of data is 6.133 and the variance is 17.8471, that is, it comes from a population with overdispersion. The GDP distribution is an ideal choice for modeling this data since it is overdispersed. We compare the fit of the GDP distribution with Discrete Pareto (see [17]), Discrete Pareto type (IV) (see [8]), Discrete Burr XII (see [20]) and Discrete Lomax (see [21]) distributions. The values of estimates, the log-likelihood function (-LogL), the Kolmogrov-Smirnov (K-S) statistic, Akaike Information Criterion (AIC), Akaike Information Criterion with cor-

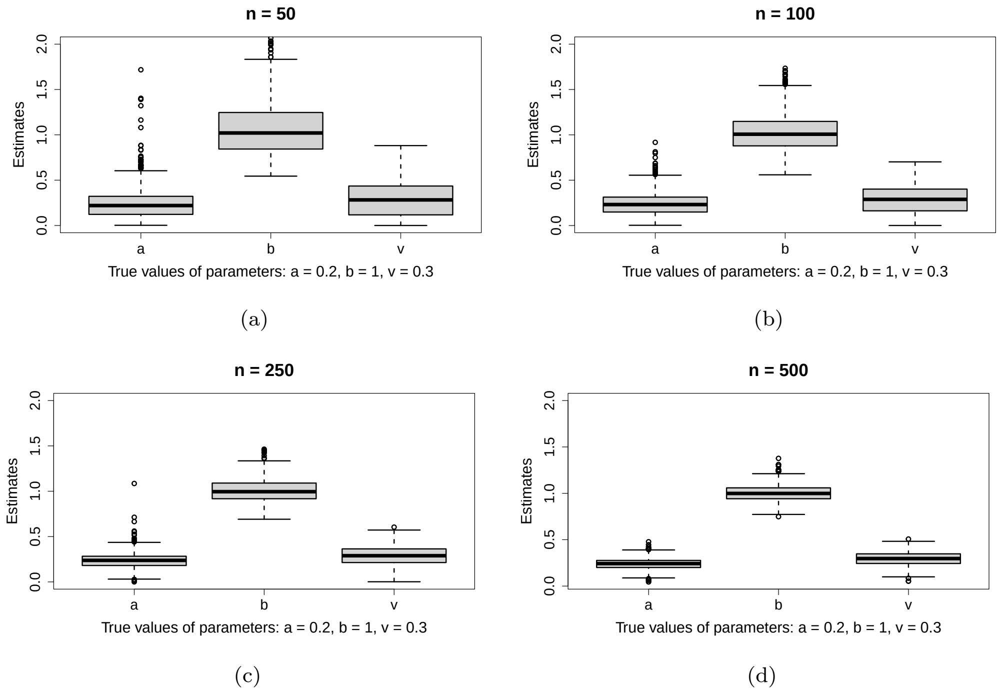
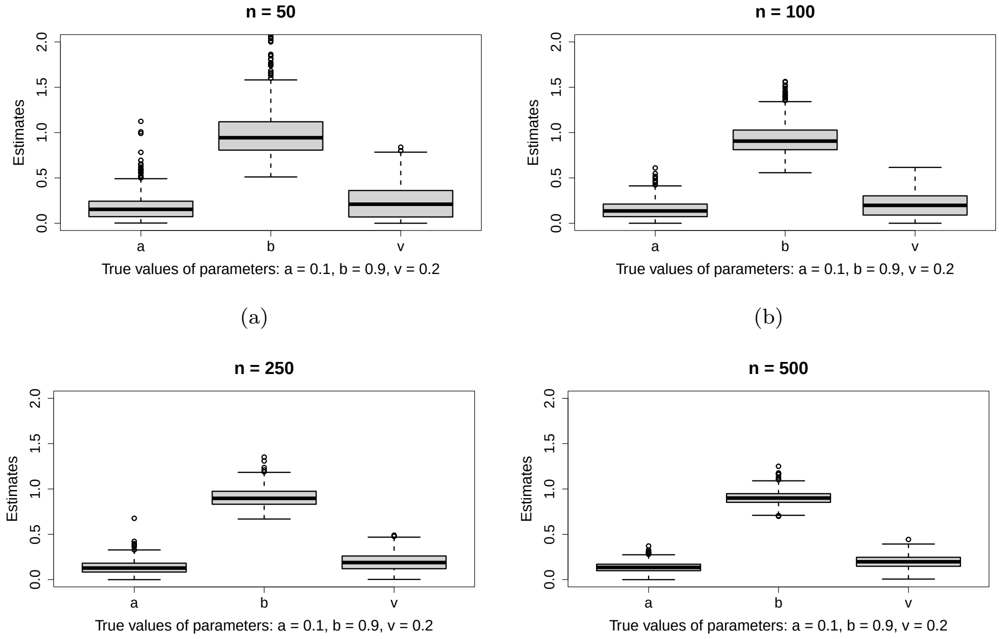


Figure 3. Box plots of the estimated values for parameters a=0.2, b=1.0 and v=0.3

Parameter	Sample size(n)	Bias	MSE	MRE	CP
	50	0.0742	0.0238	1.0901	0.952
a=0.1	100	0.0489	0.0114	0.8113	0.970
	250	0.0364	0.0063	0.6109	0.971
	500	0.0356	0.0043	0.5222	0.975
	50	0.0956	0.0937	0.2259	0.956
b=0.9	100	0.0299	0.0283	0.1424	0.962
	250	0.0042	0.0107	0.0922	0.974
	500	0.0018	0.0053	0.0640	0.985
	50	-0.0318	0.0344	0.7649	0.833
v=0.2	100	-0.0074	0.0191	0.5750	0.861
	250	0.0053	0.0097	0.4053	0.897
	500	-0.0031	0.0053	0.2934	0.927

Table 3. Simulation results for parameters a=0.1, b=0.9 and v=0.2



(a) (b) (c) (d)

Figure 4. Box plots of the estimated values for parameters a=0.1, b=0.9 and v=0.2

rection (AICc), Bayesian Information Criterion (BIC) and Hannon-Quinn Information Criterion (HQIC) are calculated for the five distributions in order to verify which distribution fits better to the data.
To address the identifiability, we computed bootstrap confidence intervals (CI) based on 1,000 resamples for the parameter estimates of each distribution. These intervals provide a robust measure of uncertainty and is valuable for discrete over-dispersed count data, where analytical confidence intervals may be unreliable or difficult to derive. These CIs allows for a more reliable assessment of both the model fit and the precision of the estimated parameters.
The summary of data set is given below:

Table 4. Summary Statistics of the Dataset

Mean	**Variance**	**Median**	**Skewness**	**Kurtosis**
6.1333	17.8471	6.0000	0.7877	3.0412

The Table 5 shows that among all competing discrete distributions, the GDP model is chosen as the best model with the lowest -LogL, AIC, BIC, AICc, HQIC. Moreover, according to the p value, 0.153 of the K-S test statistic shows GDP is the better model.

6. Conclusions

A new flexible discrete model is introduced which is a generalized version for discrete Pareto models. Some distributional properties are addressed. This distribution presents a competitive alternative to other discrete Pareto models currently available.

Table 5. Parameter estimates and goodness of fit for various models fitted to the dataset.

Model	MLEs	Bootstrap CI	-LogL	AIC	BIC	AICc	HQIC	K-S	p-value
DP	$\hat{p} = 0.5862$	(0.5655-0.6053)	315.3311	632.6622	635.1620	632.7077	633.6703	0.4439	0.027
DL	$\hat{\lambda}= 0.0056$	(0.0049-0.0064)	263.4498	530.8996	535.8992	531.0375	532.9157	0.2724	0.000
	$\hat{\alpha}$ =27.5889	(27.0127-27.6466)							
DP IV	$\hat{\theta}$=0.7870	(0.6208-1.2528)	300.0187	606.0374	613.5368	606.3165	609.0616	0.5648	0.010
	$\hat{\sigma} = 0.9521$	(0.7201-1.8339)							
	$\hat{\gamma} = 0.5210$	(0.2612-0.7424)							
DB XII	$\hat{\beta}$ =0.5274	(0.4145-0.6105)	345.1003	696.2006	703.7000	696.4797	699.2248	0.4070	0.00
	$\hat{\gamma}$ =1.3204	(1.0790-2.4434)							
	$\hat{c}$ =0.5628	(0.5040-0.7314)							
GDP	$\hat{a}$= 0.1100	(0.0929-0.2041)	245.9820	499.964	509.963	500.4346	503.9963	0.1194	0.153
	$\hat{b}$ =1.8017	(1.5548-2.2713)							
	$\hat{v}$ =0.0036	(0.0019-0.0954)							
	$\hat{\beta} = 3.4488$	(2.2898-6.5112)							

The GDP distribution has four parameters, which raises potential concerns about identifiability, especially with small sample sizes. But in case of GDP, the likelihood function has a unique maximum for distinct parameter values, provided that the data contain sufficient variability across the support. The effectiveness of the GDP distribution in modeling the number of warts observed after immunotherapy clearly shows that it is a candidate model to be used in medical field. Thus, the GDP distribution distribution shows promise for modelling over- dispersed counts.

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